BOUNDS ON PERMANENTS, AND THE NUMBER OF 1-FACTORS AND 1-FACTORIZATIONS OF BIPARTITE GRAPHS

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Abstract. We give a survey of some recent developments on bounds for permanents (Falikman-Egorychev, Voorhoeve, Bang, Brègman), and show some related results on counting 1-factors (perfect matchings), 1-factorizations (edge-colourings), and eulerian orientations of graphs.

1. INTRODUCTION.

The *permanent* of a square matrix $A = (a_{ij})_{i,j=1}^{n}$ is given by:

(1) $$\text{per}A = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i\sigma(i)},$$

where S_n denotes the collection of all permutations of $\{1,\ldots,n\}$.

Despite its appearance as the simpler twin-brother of the determinant function, the permanent turns out to be much less tractable. Whereas a determinant can be calculated quickly (in polynomial time, with Gaussian elimination), determining the permanent is difficult ("number-P-complete" - see Valiant [27]). As yet, its algebraic behaviour appeared to a large extent unmanageable, and its algebraic relevance moderate.

Most interest in permanents came from the famous Van der Waerden conjecture on the minimum permanent of doubly stochastic matrices (see below). This conjecture was unsolved for more than fifty years, which, as contrasted with its simple form, also contributed to the image of intractability of permanents. Recently, Falikman and Egorychev were able to prove this conjecture, using a classical inequality of Alexandroff and Fenchel. The proof with eigenvalue techniques also revealed some unexpected nice algebraic behaviour of the permanent function.

In fact, lower and upper bounds form a field where a large part of the successes in controlling permanents have been obtained, also by the work of, e.g., Bang, Brègman and Voorhoeve. In this paper we discuss some of the bounds for the permanent function, and for the related numbers of 1-factors and 1-factorizations of bipartite graphs. Especially, we survey some recent work in this field.

The book by Minc [21] gives an excellent survey of what is known on permanents until 1978. Van Lint [15] gave a survey of bounds on permanents known in 1974. For some more historical remarks, see Van Lint [17].

In this introduction we first give a brief survey.

Van der Waerden's conjecture. In 1926 Van der Waerden [30] posed the following conjecture: if A is a doubly stochastic matrix of order n, then

(2) $$\text{per}A \geq n!/n^n,$$

and equality only holds for $A = \frac{1}{n}J$ (J being the all-one matrix). A matrix is *doubly stochastic* if it is nonnegative and all row and column sums are 1.

As the permanent function is not convex, the Kuhn-Tucker theory (Lagrange multipliers) yields only necessary conditions for the doubly stochastic matrices minimizing the permanent. The conjecture raised a stream of research, especially during the last twenty years. In 1978, as a prelude, the lower bound of e^{-n} was proved by Bang [2] and Friedland [10], which bound is asymptotically equal to Van der Waerden's conjectured lower bound $n!/n^n$, by Stirling's formula. Ultimately in 1979 and 1980, Falikman [8] and Egorychev [6] published proofs of Van der Waerden's conjecture.

The basis for their proofs is a permanent inequality, which is a special case of an inequality for "mixed volumes" of convex bodies, found in the thirties by Fenchel [9] and Alexandroff [1] (cf. Busemann [5]). Let B be an $n\times(n-2)$-matrix, and let x and y be column vectors of length n. If B and x are nonnegative, then

(3) $$\text{per}^2(B,x,y) \geq \text{per}(B,x,x).\text{per}(B,y,y).$$

(This can be seen to be equivalent to: the function $x \longmapsto \sqrt{\text{per}(B,x,x)}$ is concave on the nonnegative orthant.) The inequality (3) can be proved directly with an interesting eigenvalue technique ([12],[16]).

On the other hand, Marcus and Newman [19] and London [18] had shown that if A is a *minimizing* matrix (i.e. a doubly stochastic matrix minimizing the permanent), then $\text{per}A_{ij} \geq \text{per}A$ for each $(n-1)\times(n-1)$-minor A_{ij} of A. Hence, if (B,x,y) is a minimizing matrix, then $\text{per}(B,x,y) \leq \text{per}(B,x,x)$ and $\text{per}(B,x,y) \leq \text{per}(B,y,y)$ (as we can expand these permanents by the last columns, just like determinants, but without sign problems). Therefore, by (3), we have $\text{per}(B,x,y) = \text{per}(B,x,x) = \text{per}(B,y,y)$. This implies

(4) $$\text{per}(B,\tfrac12x+\tfrac12y,\tfrac12x+\tfrac12y) = \tfrac14\text{per}(B,x,x)+\tfrac12\text{per}(B,x,y)+\tfrac14\text{per}(B,y,y) = \text{per}(B,x,y)$$

(using the fact that the permanent is linear in the columns). Since the matrix $(B,\frac12x+\frac12y,\frac12x+\frac12y)$ is doubly stochastic, by (4) it is minimizing again. If we assume that we have chosen the matrix (B,x,y) so that the sum of its squared components (i.e., $\text{Tr}((B,x,y)^T(B,x,y))$) is as small as possible, it follows that $x=y$ (as $\text{Tr}((B,\frac12x+\frac12y,\frac12x+\frac12y)^T(B,\frac12x+\frac12y,\frac12x+\frac12y)) \leq \text{Tr}((B,x,y)^T(B,x,y))$ with equality iff $x=y$). As the columns x and y were chosen arbitrarily, we know that all columns of (B,x,y) are equal, that is, it is $\frac{1}{n}J$.

By extending these methods Egorychev proved that $\frac{1}{n}J$ is the only minimizing matrix. In Section 2 we describe a complete proof of Van der Waerden's conjecture, where we have benefitted by the presentations of Knuth [12] and Van Lint [16,17].

Permanents combinatorially. The permanent can be put in a more combinatorial context as follows. For natural numbers k and n, denote

(5) Λ_n^k = the set of all nonnegative integral $n\times n$-matrices with all line sums equal to k

(*lines* are rows and columns). Then Falikman and Egorychev's lower bound is equivalent to:

(6) if $A \in \Lambda_n^k$ then $\text{per}A \geq (\frac{k}{n})^n n!$.

Indeed, if $A\in\Lambda_n^k$, $\frac{1}{k}A$ is doubly stochastic, and hence $\text{per}A = k^n\text{per}(\frac{1}{k}A) \geq k^n n!/n^n$. Conversely, any rational doubly stochastic matrix of order n is equal to $\frac{1}{k}A$ for some k and some $A\in\Lambda_n^k$. Then (6) gives $\text{per}(\frac{1}{k}A) = k^{-n}\text{per}A \geq n!/n^n$. So $n!/n^n$ is a lower bound for rational doubly stochastic matrices, and hence, by continuity, it is a lower bound for all doubly stochastic matrices.

To obtain a more combinatorial interpretation, if A is in Λ_n^k, we can construct the bipartite graph G with vertices $v_1,\ldots,v_n,w_1,\ldots,w_n$, connecting v_i and w_j by a_{ij} (possibly parallel) edges. Then G is k-regular, and the permanent of A is equal to the number of perfect matchings in G.

In 1968, Erdös and Rényi [7] published the following conjecture, weaker than Van der Waerden's conjecture:

(7) there is an $\varepsilon > 0$ such that if $A \in \Lambda_n^k$ with $k \geq 3$, then $\text{per}A \geq (1+\varepsilon)^n$.

This conjecture is implied by Van der Waerden's conjecture through (6), as $(k/n)^n n! \geq (k/e)^n$ by Stirling's formula.

The Erdös-Rényi conjecture was proved in 1978 independently by Voorhoeve [29] and by Bang [2] and Friedland [10]. As mentioned before, Bang and Friedland showed that $\text{per}A \geq e^{-n}$ for each doubly stochastic matrix A of order n, and hence $\text{per}A \geq (k/e)^n$ for each $A \in \Lambda_n^k$. This shows (7).

Voorhoeve showed:

(8) if $A \in \Lambda_n^3$ then $\text{per}A \geq (\frac{4}{3})^n$.

In other words, any 3-regular bipartite graph with 2n vertices has at least $(4/3)^n$ perfect matchings. Or: if A is a doubly stochastic matrix of order n, with all components a multiple of 1/3, then $\text{per}A \geq (4/9)^n$. Asymptotically, for $n \to \infty$, this is better than Falikman and Egorychev's and Bang and Friedland's lower bounds ($(3/e)^n$). The best lower bound for permanents of matrices in Λ_n^3 found before Voorhoeve's result was 3n-2 (Hartfiel and Crosby [11]). With König's theorem (see Remark 1 below) (8) implies that $\text{per}A \geq (4/3)^n$ for all $A \in \Lambda_n^k$, $k \geq 3$, and hence the Erdös-Rényi conjecture follows.

In [26] it has been shown that the ground number 4/3 in (8) is best possible. More generally, let f(k) be the highest possible number such that $\text{per}A \geq f(k)^n$ for all $A \in \Lambda_n^k$. Then

(9) $$f(k) \leq \frac{(k-1)^{k-1}}{k^{k-2}}.$$

Note that by Bang's result, $f(k) \geq k/e$, and by Voorhoeve's result, $f(3) \geq 4/3$. The latter bound combined with (9) gives $f(3) = 4/3$. Trivially we have $f(1) = f(2) = 1$. It is conjectured in [26] that equality holds in (9) for every k. That is:

(10) (Conjecture) if $A \in \Lambda_n^k$ then $\operatorname{per}A \geq \left(\frac{(k-1)^{k-1}}{k^{k-2}}\right)^n$.

This conjecture would give a bound asymptotic for k fixed and $n \to \infty$, while Falikman and Egorychev's lower bound, in the form (6), is asymptotic for n fixed, $k \to \infty$. Conjecture (10) implies a better lower bound for permanents of doubly stochastic matrices with all components being a multiple of 1/k.

Voorhoeve's method consists of a clever induction trick, which it is tempting to generalize to values of $k \geq 4$. However, in this direction no significant progress has been made as yet.

For a more extensive discussion of Voorhoeve's result and best lower bounds, see Section 3.

Bang's method and edge-colourings. The method of Bang [2] gives rise to some further graph-theoretic considerations.

Suppose you have given the first lesson of a course on graph theory. You have explained Euler's result on the existence of eulerian orientations, and you have given the definitions of regular and bipartite graphs, and of perfect matchings. Now as homework you ask: show that each 64-regular bipartite graph has a perfect matching. Is this a reasonable question for your students, whom you do not expect to discover for themselves the König-Hall theorem?

Yes, it is. They know that the 64-regular bipartite graph has an eulerian orientation. By deleting the edges oriented from the "red" points to the "blue" points, and by forgetting the orientation of the other edges, we are left with a 32-regular bipartite graph. By the same reasoning this 32-regular graph has a 16-regular spanning subgraph. And so on, until we have a 1-regular spanning subgraph, which is a perfect matching.

This idea can be extended from the *existence* of perfect matchings to *counting* perfect matchings, and also to counting 1-factorizations of regular bipartite graphs ([24]). This last can be seen as the graph-theoretic interpretation of the ideas, in matrix language, of Bang, which have led to his lower bound e^{-n}.

It also leads to the following. In [24] it is conjectured that if G is a k-regular bipartite graph with 2n points, then

(11) (Conjecture) G has at least $(k!^2/k^k)^n$ 1-factorizations.

By an averaging argument it can be shown that the ground number in (11), as a function of k, cannot be higher. Moreover, using the ideas described above, and using Voorhoeve's lower bound, it can be shown that (11) is true if k has no other prime factors than 2 and 3.

These results are described more extensively in Section 4.

Brègman's upper bound. Now we turn to upper bounds. It is easy to see that the maximum permanent over the doubly stochastic matrices is 1, which is attained, exclusively, by the permutation matrices. Similarly, the maximum permanent over matrices in Λ_n^k is equal to k^n.

The problem becomes more difficult if we go over to a further discretization. In 1963, Minc [20] posed as a conjecture:

(12) if A is a square (0,1)-matrix of order n, with row sums $r_1,\dots,r_n$, then

$$\text{per}A \le \Pi_{i=1}^n \; r_i!^{1/r_i}.$$

In 1973, Brègman [4] found a proof for this conjecture, using ideas from convex programming, and some theory of doubly stochastic matrices. In [23] a shorter proof was given, using elementary counting and the convexity of the function xlogx.

Note that (12) implies that

(13) if $A \in \Lambda_n^k$ and A is (0,1), then $\text{per}A \le \left(k!^{1/k}\right)^n$.

The ground number here can be easily seen to be asymptotically best possible (for fixed k).

The proof of Brègman's upper bound is given in Section 5.

Eulerian orientations. Finally, as a further illustration of the methods, we consider bounds for the number of eulerian orientations of undirected graphs. Let G = (V,E) be a loopless, 2k-regular undirected graph, with |V| = n and |E| = m. Let $\varepsilon(G)$ denote the number of eulerian orientations of G. Let B be the n×m-incidence matrix of G, and let A be the m×m-matrix obtained from B by repeating each row k times. Then one easily sees:

(14) $\varepsilon(G) = \frac{\text{per}A}{k!^n}.$

Now it can be shown that

(15) $\left(2^{-k}\binom{2k}{k}\right)^n \leq \varepsilon(G) \leq \sqrt{\binom{2k}{k}}\ ^n.$

The upper bound can be derived straightforwardly from Brĕgman's bound (12) using (14). The lower bound in (15) is better than the one derived with (14) from the conjectured lower bound (10).

It can be shown moreover that the ground numbers in (15) are best possible. These results are described further in Section 6.

Throughout this paper, n denotes the order of the matrix in question. Furthermore, if the matrix A is given, A_{ij} denotes the minor of A obtained by deleting the i-th row and the j-th column of A.

REMARK 1. We here remark the following well-known facts.

(16) *Doubly stochastic matrices minimizing the permanent exist.*

This follows of course from the compactness of the set of doubly stochastic matrices, and from the continuity of the permanent function.

(17) *Each doubly stochastic matrix is a convex linear combination of permutation matrices.*

This result of Birkhoff [3] and Von Neumann [22] can be seen by induction on n. It suffices to show that each vertex of the polytope of doubly stochastic matrices is a convex linear combination of permutation matrices (and hence is a permutation matrix itself). Let $A = (a_{ij})_{i,j=1}^n$ be a vertex of this polytope. Then n^2 linearly independent inequalities in the system: $a_{ij} \geq 0$ $(i,j=1,\ldots,n)$, $\sum_i a_{ij} = 1$ $(j=1,\ldots,n)$, $\sum_j a_{ij} = 1$ $(i=1,\ldots,n)$, are satisfied with equality. So A has at least n^2-2n+1 zeros, and hence at least one row has n-1 zeros. So $a_{ij} = 1$ for some i,j. Then A_{ij} is doubly stochastic again, and by the induction hypothesis, it is a convex linear combination of permutation matrices of order n-1. Therefore, A itself is a convex linear combination of permutation matrices of order n.

(17) implies:

(18) $\mathrm{per}A > 0$ *if* A *is a doubly stochastic matrix;* $\mathrm{per}A \geq 1$ *if* A *is in* Λ_n^k.

The second assertion is equivalent to a result of König [14]: each k-regular bipartite graph has a perfect matching. So for each $A \in \Lambda_n^k$ there exists an $A' \in \Lambda_n^{k-1}$ with $A' \leq A$ ($\leq$ component-wise). Inductively it implies that each k-regular bipartite graph has a k-edge colouring, which is another theorem of König [13].

2. FALIKMAN AND EGORYCHEV'S PROOF OF THE VAN DER WAERDEN CONJECTURE.

Van der Waerden's conjecture (2) was proved by Falikman [8] and Egorychev [6] (cf. Knuth [12] and Van Lint [16,17]). The ingredients are two results, the first one being a special case of an inequality for "mixed volumes" of convex bodies, due to Fenchel [9] and Alexandroff [1] (cf. Busemann [5]).

THEOREM 1 (Alexandroff-Fenchel permanent inequality). *If* B *is a nonnegative* $n\times(n-2)$*-matrix,* x *and* y *are column vectors of length* n*, and* $x \geq 0$*, then*

(19) $$\mathrm{per}^2(B,x,y) \geq \mathrm{per}(B,x,x).\mathrm{per}(B,y,y).$$

If B *and* x *are strictly positive, equality holds in* (19) *if and only if* $y = \lambda x$ *for some* λ.

PROOF. The proof is by induction on n, the case n=2 being easy. Suppose the theorem has been proved for n-1. To prove (19), by continuity we may assume that all components of B and x are positive. Define the matrix $Q = (q_{ij})_{i,j=1}^n$ by:

(20) $$q_{ij} = \mathrm{per}(B,e_i,e_j),$$

where e_i and e_j denote the i-th and the j-th column standard basis vectors. So $\mathrm{per}(B,x,y) = x^TBy$.

I. We first show that Q is nonsingular with exactly one positive eigenvalue (i.e., it defines a "Lorentz space"). To see that Q is nonsingular, assume that $Qc = 0$, that is:

(21) $\quad \mathrm{per}(B,c,e_j) = 0$

for $j = 1,\ldots,n$. Let $B = (C,z)$, where z is the last column of B (so C is an $n\times(n-3)$-matrix). Then for each $j = 1,\ldots,n$:

(22) $\quad 0 = \mathrm{per}^2(C,z,c,e_j) \geq \mathrm{per}(C,z,z,e_j).\mathrm{per}(C,c,c,e_j).$

The equality here follows from (21), and the inequality from our induction hypothesis: as e_j is the j-th standard basis vector, the matrices in (22) can be replaced by their (j,n)-th minors.

Since $\mathrm{per}(C,z,z,e_j) > 0$ (as C and z are positive), (22) gives that $\mathrm{per}(C,c,c,e_j) \leq 0$. As from (21) $\mathrm{per}(C,z,c,e_j) = 0$ for all j, we know:

(23) $\quad 0 = \mathrm{per}(C,c,c,z) = \sum_{j=1}^{n} z_j \mathrm{per}(C,c,c,e_j) \leq 0.$

As z is positive, (23) implies that $\mathrm{per}(C,c,c,e_j) = 0$ for all j. Hence the inequality in (22) holds with equality, for all j, and therefore, from the induction hypothesis, $c = \lambda z$ for some λ. If $\lambda \neq 0$ then $0 = \mathrm{per}(B,c,e_j) = \lambda\mathrm{per}(B,z,e_j) \neq 0$ (as B and z are positive), which is a contradiction. So $\lambda = 0$ and hence $c = 0$. Concluding $Qc = 0$ implies $c = 0$, and so Q is nonsingular.

Now, for each real number μ, let the matrix Q_μ be defined by:

(24) $\quad Q_\mu = (\mathrm{per}(\mu B+(1-\mu)J,e_i,e_j))_{i,j=1}^{n}$

(here J denotes the all-one $n\times(n-2)$-matrix). So $Q_1 = Q$. Since $\mu B+(1-\mu)J$ is a positive matrix for $0 \leq \mu \leq 1$, we know by the above that Q_μ is nonsingular for $0 \leq \mu \leq 1$. For $\mu = 0$, Q_μ is a matrix with zero diagonal and with all off-diagonal components equal to $(n-2)!$, and so it has exactly one positive eigenvalue. Therefore, as the shift of the spectrum of Q_μ is continuous in μ, also for $\mu = 1$ the matrix $Q_\mu = Q$ has exactly one positive eigenvalue.

II. We now prove the theorem. The inequality (19) is equivalent to:

(25) $\quad (x^TQy)^2 \geq (x^TQx).(y^TQy).$

This inequality holds trivially with equality if x and y are linearly dependent. If x and y are linearly independent, the (2-dimensional) linear hull of x and y intersects the ((n-1)-dimensional) linear hull of the eigenvectors of Q with negative eigenvalue in a nonzero vector (as Q has n-1 negative eigenvalues). Therefore

(26) $$(\lambda x+\mu y)^T Q(\lambda x+\mu y) < 0$$

for some λ,μ not both zero. Since $x^T Qx = \text{per}(B,x,x) > 0$ (as $x > 0$), we know that $\mu \neq 0$. We may assume $\mu = 1$. Then the left hand side of (26) becomes a quadratic polynomial in λ, with positive main coefficient $x^T Qx$, and at least one negative value. Hence its discriminant is positive, which means that (25) holds, with strict inequality. □

A second ingredient for the proof of Van der Waerden's conjecture is a theorem due to Marcus and Newman [19] and London [18].

<u>THEOREM 2</u>. *If* A *is a doubly stochastic matrix minimizing the permanent, then* $\text{per}A_{ij} \geq \text{per}A$ *for each minor* A_{ij} *of* A.

<u>PROOF</u>. Let A be a minimizing matrix of order n. Consider the directed bipartite graph G with vertices $u_1,\ldots,u_n,v_1,\ldots,v_n$, and with arcs:

(27) (i) (u_i,v_j) iff $\text{per}A_{ij} \leq \text{per}A$;
(ii) (v_j,u_i) iff $a_{ij} > 0$ and $\text{per}A_{ij} \geq \text{per}A$.

Assume that, say, $\text{per}A_{11} < \text{per}A$. We first show that then the arc (u_1,v_1) of G is not contained in any directed cycle of G. For suppose that C is such a cycle. Let $\varepsilon > 0$, and

(28) (i) replace a_{ij} by $a_{ij}+\varepsilon$ if (u_i,v_j) belongs to C,
(ii) replace a_{ij} by $a_{ij}-\varepsilon$ if (v_j,u_i) belongs to C.

Let A_ε be the matrix arising in this way. Now $\text{per}A_\varepsilon$ is a polynomial in ε, and:

(29) $$\text{per}A_\varepsilon = \text{per}A + \varepsilon\Big(\sum_{(u_i,v_j)\in C}\text{per}A_{ij} - \sum_{(v_j,u_i)\in C}\text{per}A_{ij}\Big) + O(\varepsilon^2) \qquad (\varepsilon\downarrow 0).$$

The coefficient of ε in (29) is negative, by (27) and as $\text{per}A_{11} < \text{per}A$ (the first summation is strictly smaller than $\frac{1}{2}|C|\text{per}A$, and the second summation is at least $\frac{1}{2}|C|\text{per}A$). So by choosing ε small enough A_ε is doubly stochastic with $\text{per}A_\varepsilon < \text{per}A$, contradicting that A is minimizing.

So the arc (u_1,v_1) is not contained in any directed cycle. Let, say, $v_1,\ldots,v_k,u_{t+1},\ldots,u_n$ be the points of G which can be reached by a directed path from v_1. So $k,t \ge 1$, and G has no arcs (u_i,v_j) with $i \ge t+1$ and $j \ge k+1$, nor arcs (v_j,u_i) with $j \le k$ and $i \le t$. That is:

(30) (i) if $i \ge t+1$ and $j \ge k+1$ then $\text{per}A_{ij} > \text{per}A$;
(ii) if $i \le t$ and $j \le k$ then $a_{ij} = 0$ or $\text{per}A_{ij} < \text{per}A$.

Now:

(31) $$\begin{aligned}(n-k-t)\text{per}A &= \textstyle\sum_{i>t}\sum_j a_{ij}\text{per}A_{ij} - \sum_{j\le k}\sum_i a_{ij}\text{per}A_{ij} = \\ &= \left(\textstyle\sum_{i>t}\sum_{j>k}a_{ij}\text{per}A_{ij} - \sum_{i\le t}\sum_{j\le k}a_{ij}\text{per}A_{ij}\right) \ge \left(\sum_{i>t}\sum_{j>k}a_{ij} - \sum_{i\le t}\sum_{j\le k}a_{ij}\right)\text{per}A = \\ &= \left(\textstyle\sum_{i>t}\sum_j a_{ij} - \sum_{j\le k}\sum_i a_{ij}\right)\text{per}A = (n-k-t)\text{per}A.\end{aligned}$$

Here the inequality follows from (30). The equalities follow from $\sum_j a_{ij}=1$ and $\sum_j a_{ij}\text{per}A_{ij} = \text{per}A$ for all i (and similarly for j), and by crossing out equal terms in the summations.

Since the first and the last term in (31) are equal, the inequality is an equality. Hence, by (30), $a_{ij} = 0$ if $i \le t$, $j \le k$ or if $i > t$, $j > k$. Therefore, all terms in (31) are zero, and hence $n = k+t$.

Since $k,t \ge 1$ and $n = k+t$, it follows that $k,t \le n-1$. Hence from (30), $\text{per}A_{nn} > \text{per}A > 0$. So there is a permutation σ of $\{1,\ldots,n-1\}$ with $a_{i\sigma(i)} > 0$ for $i = 1,\ldots,n-1$. As $k > (n-t)-1$ this implies that $a_{ij} > 0$ for at least one pair of $i \le t$, $j \le k$, contradicting what we showed above. □

(Alternatively, Theorem 2 can be proved using Kuhn-Tucker theory.)

Combining Theorems 1 and 2 gives the theorem of Falikman and Egorychev.

THEOREM 3 (Falikman-Egorychev theorem). *If* A *is a doubly stochastic matrix of order* n, *then* $\text{per}A \ge n!/n^n$.

PROOF. We first show that if $A = (B,x,y)$ is a doubly stochastic matrix minimizing the permanent (where x and y are the last two columns of A), then

(32) $$\text{per}(B,x,y) = \text{per}(B,x,x) = \text{per}(B,y,y).$$

Indeed, by Theorem 2,

$$\text{(33)} \qquad \mathrm{per}(B,x,x) = \sum_i x_i \mathrm{per}(B,x,e_i) \geq \mathrm{per}(B,x,y)\sum_i x_i = \mathrm{per}(B,x,y).$$

Similarly, $\mathrm{per}(B,y,y) \geq \mathrm{per}(B,x,y)$. On the other hand, by Theorem 1, $\mathrm{per}^2(B,x,y) \geq \mathrm{per}(B,x,x)\mathrm{per}(B,y,y)$. Since $\mathrm{per}(B,x,y) > 0$ (cf. (18)), it follows that $\mathrm{per}(B,x,y) = \mathrm{per}(B,x,x) = \mathrm{per}(B,y,y)$.

(32) implies that:

$$\text{(34)} \qquad \mathrm{per}(B,\tfrac12x+\tfrac12y,\tfrac12x+\tfrac12y) = \tfrac14\mathrm{per}(B,x,x)+\tfrac12\mathrm{per}(B,x,y)+\tfrac14\mathrm{per}(B,y,y) = \mathrm{per}(B,x,y).$$

Since $(B,\tfrac12x+\tfrac12y,\tfrac12x+\tfrac12y)$ is doubly stochastic again, it is again minimizing.

Now suppose we have chosen A such that $\sum_{i,j} a_{ij}^2 = \mathrm{Tr}A^TA$ is as small as possible (this is possible by compactness). Assume $A \neq (1/n)J$. Without loss of generality, $A = (B,x,y)$ with $x \neq y$. By the above, the matrix $A' := (B,\tfrac12x+\tfrac12y,\tfrac12x+\tfrac12y)$ is minimizing again. However, $\mathrm{Tr}(A'^TA') < \mathrm{Tr}(A^TA)$ (as $x \neq y$), contradicting our assumption.

Therefore, $A = (1/n)J$, and $\mathrm{per}A = n!/n^n$. □

Extension of these arguments gives the uniqueness of $(1/n)J$ as a minimizing matrix. Suppose there exists a doubly stochastic matrix $A \neq (1/n)J$ with $\mathrm{per}A = n!/n^n$. Choose such A with as few zero components as possible.

If at least n-1 columns of A are strictly positive, we can assume that $A = (B,x,y)$ with $B > 0$, $x > 0$ and $x \neq y$. Then from (32) it follows that we have equality in (19). Hence by Theorem 1, $y = \lambda x$ for some λ. As A is doubly stochastic, we have $\lambda = 1$ and $x = y$, contradicting our assumption.

If A has at most n-2 strictly positive columns, we can assume that $A = (B,x,y)$ is such that not all columns of B are positive, and such that y has a zero in at least one coordinate place where x is positive. Then by (34) $(B,\tfrac12x+\tfrac12y,\tfrac12x+\tfrac12y)$ is again a minimizing matrix, distinct from $(1/n)J$, but with fewer zeros than A, contradicting our choice of A.

3. VOORHOEVE'S BOUND AND BEST LOWER BOUNDS.

Erdös and Rényi [7] posed in 1968 the following conjecture, weaker than Van der Waerden's conjecture: there exists an $\varepsilon > 0$ such that if $A \in \Lambda_n^k$ with $k \geq 3$ then $\mathrm{per}A \geq (1+\varepsilon)^n$. We recall that Λ_n^k denotes the set of nonnegative integral n×n-matrices with all line sums equal to k.

Erdös and Rényi's conjecture was proved independently by Voorhoeve [29] and by Bang [2] and Friedland [10]. The latter two showed that $\mathrm{per}A \geq e^{-n}$ for each doubly stochastic matrix of order n. Hence $\mathrm{per}A = k^n \mathrm{per}((1/k)A) \geq (k/e)^n$ for A in Λ_n^k. For a derivation of this result, see Section 4.

In this section we focus on Voorhoeve's result, which says that $\mathrm{per}A \geq (4/3)^n$ for each $A \in \Lambda_n^3$. This improves lower bounds found earlier considerably the best one being $\mathrm{per}A \geq 3n-2$ for $A \in \Lambda_n^3$ (Hartfiel and Crosby [11]).

The trick of Voorhoeve consists of considering the collection:

(35) $\tilde{\Lambda}_n^3$:= the collection of nonnegative integral n×n-matrices with row sums 2,3,...,3 and column sums 2,3,...,3.

He showed that also for matrices A in $\tilde{\Lambda}_n^3$ one has $\mathrm{per}A \geq (4/3)^n$. This stronger result turned out to be the key to applying induction.

<u>THEOREM 4</u> (Voorhoeve's bound). *If* $A \in \Lambda_n^3$ *then* $\mathrm{per}A \geq (4/3)^n$.

<u>PROOF</u>. It is shown that $\mathrm{per}A \geq (4/3)^n$ for $A \in \tilde{\Lambda}_n^3$ by induction on n. This implies the theorem, as if $A \in \Lambda_n^3$ and B arises from A by decreasing one positive entry of A by one, then $B \in \tilde{\Lambda}_n^3$ and $\mathrm{per}A \geq \mathrm{per}B \geq (4/3)^n$.

So let $A \in \tilde{\Lambda}_n^3$. Without loss of generality the first row and the first column both have sum 2. There are the following four cases, possibly after permuting the columns of A (a,b and c denote column vectors of length n-1).

(36) $$\mathrm{per}A = \mathrm{per}\begin{pmatrix} 0 & 1 & 1 & 0\ldots0 \\ a & b & c & D \end{pmatrix} \overset{1}{=} \mathrm{per}(a,b,D)+\mathrm{per}(a,c,D) \overset{2}{=} \mathrm{per}(a,b+c,D) \overset{3}{=} \frac{1}{3}(\mathrm{per}(a,d_1,D)+\mathrm{per}(a,d_2,D)+\mathrm{per}(a,d_3,D)+\mathrm{per}(a,d_4,D)) \overset{4}{\geq} \frac{1}{3}.4\left(\frac{4}{3}\right)^{n-1} = (4/3)^n.$$

(Explanation: [1] follows by expanding the permanent by the upper row; [2] follows as the permanent is linear in the columns; [3] the components of b+c add up to 4; hence we can write $b+c = \frac{1}{3}(d_1+d_2+d_3+d_4)$ with d_1,d_2,d_3,d_4 nonnegative integral column vectors, each with column sum 3; [4] this inequality follows from the induction hypothesis, as each (a,d_i,D) belongs to $\tilde{\Lambda}_{n-1}^3$.)

(37) $$\mathrm{per}A = \mathrm{per}\begin{pmatrix} 0 & 2 & 0\ldots0 \\ a & b & D \end{pmatrix} \overset{5}{=} 2.\mathrm{per}(a,D) \overset{6}{\geq} 2\left(\frac{4}{3}\right)^{n-1} \geq \left(\frac{4}{3}\right)^n.$$

(Explanation:[5] expand the permanent by the upper row; [6] since (a,D) belongs to $\tilde{\Lambda}^3_{n-1}$, we can apply the induction hypothesis.)

$$\text{(38)} \qquad \text{per}A = \text{per}\begin{pmatrix} 1 & 1 & 0\ldots0 \\ 0 & 0 & D \end{pmatrix} \overset{7}{=} \text{per}(a,D)+\text{per}(b,D) \overset{8}{=} \text{per}(a+b,D) \overset{9}{=}$$
$$\tfrac{1}{2}(\text{per}(d_1,D)+\text{per}(d_2,D)+\text{per}(d_3,D)) \overset{10}{\geq} \frac{3}{2}.\left(\frac{4}{3}\right)^{n-1} \geq \left(\frac{4}{3}\right)^n.$$

(Explanation:[7] expand the permanent by the upper row; [8] as the permanent is linear in the columns; [9] the components of a+b add up to 3; write $a+b = \frac{1}{2}(d_1+d_2+d_3)$ with d_1,d_2,d_3 nonnegative integral vectors each with sum 2; [10] since each matrix (d_i,D) belongs to $\tilde{\Lambda}^3_{n-1}$, this inequality follows from the induction hypothesis.)

$$\text{(39)} \qquad \text{per}A = \text{per}\begin{pmatrix} 2 & 0\ldots0 \\ 0 & D \end{pmatrix} \overset{11}{=} 2.\text{per}D \overset{12}{\geq} 2.\text{per}D' \overset{13}{\geq} 2.\left(\frac{4}{3}\right)^{n-1} \geq \left(\frac{4}{3}\right)^n.$$

(Explanation:[11] expand the permanent by the upper row; [12] let D' arise from D by decreasing one positive entry of D by one; [13] since $D' \in \tilde{\Lambda}^3_{n-1}$, this inequality follows from the induction hypothesis.) □

By sharpening the method, Voorhoeve showed the better lower bound of $\frac{81}{32}(\frac{4}{3})^n$. However, the ground number 4/3 is best possible. This follows by taking k = 3 in the following result of [26] (cf. Wilf [31]), which is proved by an averaging argument.

<u>THEOREM 5</u>. *Let* f(k) *be the largest number such that* $\text{per}A \geq f(k)^n$ *for each* $A \in \Lambda^k_n$. *Then*

$$\text{(40)} \qquad f(k) \leq \frac{(k-1)^{k-1}}{k^{k-2}}.$$

<u>PROOF</u>. Let $P_{k,n}$ be the collection of all ordered partitions of {1,2,...,nk} into n classes of size k. So we have

$$\text{(41)} \qquad p_{k,n} := |P_{k,n}| = \frac{(nk)!}{k!^n}.$$

A *system of distinct representatives* (SDR) of a partition $\mathcal{A} = (A_1,\ldots,A_n)$ in $P_{k,n}$ is a subset S of {1,...,nk} such that $|S \cap A_i| = 1$ for i=1,...,n. Clearly, the number of SDR's of $\mathcal{A}$ is equal to k^n.

Now let $\mathcal{A} = (A_1,\ldots,A_n)$ and $\mathcal{B} = (B_1,\ldots,B_n)$ be in $P_{k,n}$. Let $s(\mathcal{A},\mathcal{B})$

denote the number of *common* SDR's of $\mathcal{A}$ and $\mathcal{B}$. Then $s(\mathcal{A},\mathcal{B})$ is equal to the permanent of the matrix $C = (c_{ij})_{i,j=1}^{n}$, where

(42) $$c_{ij} = |A_i \cap B_j| \qquad (i,j = 1,\ldots,n).$$

Indeed, if σ is a permutation of $\{1,\ldots,n\}$, then $\Pi_{i=1}^{n} a_{i\sigma(i)}$ is the number of common SDR's S containing an element in $A_i \cap B_{\sigma(i)}$, for each i. Hence

(43) $$s(\mathcal{A},\mathcal{B}) = \sum_{\sigma\in S_n} \Pi_{i=1}^{n} a_{i\sigma(i)} = \text{per}C.$$

Since $\sum_{i=1}^{n} c_{ij} = |B_j| = k = |A_i| = \sum_{j=1}^{n} c_{ij}$, we know that $C \in \Lambda_n^k$. Therefore,

(44) $$s(\mathcal{A},\mathcal{B}) \geq f(k)^n.$$

Now let $\mathcal{A} \in P_{k,n}$ be fixed. Any SDR $S = \{s_1,\ldots,s_n\}$ of $\mathcal{A}$ is an SDR of $n!p_{k-1,n}$ partitions $\mathcal{B}$ in $P_{k,n}$, as we can distribute $s_1,\ldots,s_n$ in n! ways among $B_1,\ldots,B_n$, whereas the other elements of $B_1,\ldots,B_n$ can be chosen freely. Since $\mathcal{A}$ has k^n SDR's, we find

(45) $$\sum_{\mathcal{B}\in P_{k,n}} s(\mathcal{A},\mathcal{B}) = k^n n!\, p_{k-1,n}.$$

Combining (41), (44) and (45) gives:

(46) $$f(k)^n \leq \frac{k^n n! p_{k-1,n}}{p_{k,n}} = \frac{k^n n! k!^n (nk-n)!}{(k-1)!^n (nk)!} = k^{2n}/\binom{nk}{n}.$$

By Stirling's formula, (46) implies (40). □

We conjecture that in fact the upper bound in Theorem 5 always gives the right value of f(k). This is trivially true if $k=1$ or $k=2$ (as $f(1) = f(2) = 1$), and is also true for $k=3$ by Voorhoeve's theorem (Theorem 4). At the end of the following Section we shall see some more lower bounds for f(k).

Note that the proof of Theorem 5 in fact gives (cf. (46)):

(47) $$\min_{A\in\Lambda_n^k} \text{per } A \leq k^{2n}/\binom{nk}{n}.$$

4. BANG'S LOWER BOUND AND EDGE-COLOURINGS.

We now give a proof of Bang's lower bound of e^{-n} for permanents of doubly stochastic matrices of order n. His method can be interpreted, and extended, in terms of edge-colourings, or 1-factorizations, of bipartite graphs. A k-*edge-colouring* of a bipartite graph is an ordered partition of the edge set of the graph into k classes, each class being a perfect matching. It is a well-known theorem of König [13] that each k-regular bipartite graph has at least one k-edge-colouring (see Remark 1 in Section 1). Here we consider counting them.

In [24] it is shown that if $k = 2^a 3^b$, and G is a k-regular bipartite graph with 2n vertices, then

(48) G has at least $\left(\frac{k!^2}{k^k}\right)^n$ k-edge-colourings.

Moreover it is shown that for each fixed k, the ground number in (48) is best possible. It is conjectured that (48) holds for every k. This conjecture would follow from the conjecture made in Section 3 that $f(k) = (k-1)^{k-1}/k^{k-2}$ for each k, that is, that each k-regular bipartite graph has at least $\left((k-1)^{k-1}/k^{k-2}\right)^n$ perfect matchings. We could first choose a perfect matching, delete this perfect matching, next choose a perfect matching in the remainder, and so on. Hence G would have at least

(49) $$\left(\frac{(k-1)^{k-1}}{k^{k-2}} \cdot \frac{(k-2)^{k-2}}{(k-1)^{k-3}} \cdot \ldots \cdot \frac{2^2}{3^1} \cdot \frac{1^1}{2^0}\right)^n = \left(\frac{k!^2}{k^k}\right)^n$$

k-edge-colourings.

In other words, let g(k) be the highest number such that each k-regular bipartite graph with 2n points has at least $g(k)^n$ k-edge-colourings. Then $g(k) \le k!^2/k^k$, and we have equality if $k = 2^a 3^b$. This is the content of the following two theorems, the first one being proved similarly to Theorem 5.

<u>THEOREM 6</u>. $g(k) \le k!^2/k^k$.

<u>PROOF</u>. Again, let $P_{k,n}$ and $p_{k,n}$ be as in the proof of Theorem 5. For $\mathcal{A},\mathcal{B}$ in $P_{k,n}$ denote by $c(\mathcal{A},\mathcal{B})$ the number of partitions $\mathcal{C} = (C_1,\ldots,C_k)$ of $\{1,\ldots,nk\}$ into k classes of size n such that

(50) $$|A_i \cap C_j| = |B_i \cap C_j| = 1$$

for $i = 1,\ldots,n$ and $j = 1,\ldots,k$. That is, each C_j is a common SDR for $\mathcal{A}$ and $\mathcal{B}$. It is easy to see that $c(\mathcal{A},\mathcal{B})$ is equal to the number of k-edge-colourings of the k-regular bipartite graph with vertices, say, $v_1,\ldots,v_n,w_1,\ldots,w_n$, where v_i and w_j are connected by $|A_i \cap B_j|$ edges, for $i,j = 1,\ldots,n$. In particular,

(51) $$c(\mathcal{A},\mathcal{B}) \geq g(k)^n.$$

Now let $\mathcal{A} \in P_{k,n}$ be fixed. There are $k!^n$ possible partitions $\mathcal{C} = (C_1,\ldots,C_n)$ of $\{1,\ldots,nk\}$ with $|A_i \cap C_j| = 1$ for $i = 1,\ldots,n$ and $j = 1,\ldots,k$. For each such partition, there are $n!^k$ partitions $\mathcal{B}$ in $P_{k,n}$ such that $|B_i \cap C_j| = 1$ for $i = 1,\ldots,n$ and $j = 1,\ldots,k$. So

(52) $$\sum_{\mathcal{B} \in P_{k,n}} c(\mathcal{A},\mathcal{B}) = k!^n . n!^k.$$

Combining (41), (51) and (52) gives

(53) $$g(k)^n \leq \frac{k!^{2n} n!^k}{(nk)!} .$$

By Stirling's formula, (53) implies Theorem 6. □

A special case of the idea behind the next theorem is the following. Let $G = (V,E)$ be a 2k-regular bipartite graph, with 2n points. A *k-factor* is a collection E' of edges of G such that each point is contained in exactly k edges in E'. So E' is a k-factor in G if and only if $E \setminus E'$ is a k-factor.

Now it is easy to see that the number of k-factors of G is equal to the number $\varepsilon(G)$ of eulerian orientations of G. The latter can be seen to be at least

(54) $$(2^{-k}\binom{2k}{k})^{2n}.$$

Indeed, we can replace the graph G by a graph G', by splitting each point v of G into k copies, and by distributing the 2k edges incident with v among the k copies of v, in such a way that G' will be 2-regular. Then G' trivially has an eulerian orientation, which induces an eulerian orientation in G. Moreover, each eulerian orientation in G arises in this way from an eulerian orientation in exactly $k!^{2n}$ graphs G' (as in each point of G we have to make pairs of an ingoing and an outgoing edge). Since there are exactly

(55) $$\left(2^{-k}\frac{(2k)!}{k!}\right)^{2n}$$

graphs G' in total, the number of eulerian orientations of G is at least (55) divided by $k!^{2n}$, which is (54).

With this it can be seen that any 2^t-regular bipartite graph $G = (V,E)$ on 2n points has at least

(56) $$\left(\frac{(2^t)!^2}{2^{t2^t}}\right)^n$$

2^t-edge-colourings (by Theorem 6, the ground number in (56) is best possible). This can be shown by induction on t, the case $t=0$ being trivial. By (54), G has at least

(57) $$\left(2^{-2^{t-1}}\binom{2^t}{2^{t-1}}\right)^{2n}$$

2^{t-1}-factors E'. By induction, the graphs (V,E') and (V,E\E') have at least

(58) $$\left(\frac{(2^{t-1})!^2}{2^{(t-1)2^{(t-1)}}}\right)^n$$

2^{t-1}-edge-colourings. So the number of 2^t-edge-colourings of G is at least (58) squared times (57), which is (56).

This idea is extended in Theorem 7.

THEOREM 7. *If* $g(k) = k!^2/k^k$ *for* $k=s$ *and* $k=t$, *then also for* $k=st$.

PROOF. Let $G = (V,E)$ be an st-regular bipartite graph with 2n points, with, say, $\phi(G)$ st-edge-colourings. Consider all possible graphs G' arising from G as follows. Each point of G is split into s new vertices, where each edge e of G is replaced by one new edge connecting two of the new vertices replacing the endpoints of the original edge e, in such a way that the new graph G' is t-regular. So the number of graphs G' arising in this way from G is equal to:

(59) $$\left(\frac{(st)!}{t!^s}\right)^{2n},$$

since for each point v of G we have to partition the edges incident to v into s classes of size t, which can be done in $(st)!/t!^s$ ways.

Let Π be the collection of all partitions $(E_1,\ldots,E_t)$ of the edge set of G into t classes, such that each class E_j is an s-factor of G. Now any t-edge-colouring $(E_1,\ldots,E_t)$ of a derived graph G' yields a partition in Π. Conversely, each partition in Π arises in this way from a t-edge-colouring of $s!^{2tn}$ graphs G' (as for each point v of G and for each j=1,...,t, we have to take care that the edges in E_j incident to v will go to distinct copies of v in G', which means that for each v and j there are s! possibilities).

Hence, by (59),

(60) $$|\Pi| \geq \left(\frac{(st)!}{t!^s}\right)^{2n}.g(t)^{sn}\Big/ s!^{2tn}$$

as each graph G' has at least $g(t)^{sn}$ t-edge-colourings.

Now each class E_j of a partition E in Π can be refined to an s-edge-colouring of the graph (V,E_j) in at least $g(s)^n$ ways. So E can be refined to an st-edge-colouring of G in at least $g(s)^{tn}$ ways. Therefore, the total number $\phi(G)$ of st-edge-colourings of G satisfies (using (60)):

(61) $$\phi(G) \geq |\Pi|.g(s)^{tn} \geq \left(\frac{(st)!}{t!^s.s!^t}\right)^{2n}.g(s)^{tn}.g(t)^{sn} = \left(\frac{(st)!^2}{(st)^{st}}\right)^n.$$

As this holds for each st-regular bipartite graph G with 2n points, it follows that $g(st) \geq (st)!^2/(st)^{st}$. □

This implies the following.

<u>COROLLARY 7a</u>. *If* k *has no other prime factors than* 2 *and* 3, *then any* k-*regular bipartite graph with* 2n *points has at least* $(k!^2/k^k)^n$ k-*edge-colourings. For fixed* k *this ground number is best possible.*

<u>PROOF</u>. By Theorems 6 and 7 it suffices to show that $g(2) \geq 1$ and $g(3) \geq 4/3$. The former inequality is trivial, while the latter follows from Voorhoeve's lower bound (Theorem 4) that the number of perfect matchings in a 3-regular bipartite graph with 2n points is at least $(4/3)^n$. □

From Theorem 7 one can also derive the lower bound of Bang [2] and Friedland [10].

COROLLARY 7b. *The permanent of a doubly stochastic matrix of order* n *is at least* e^{-n}.

PROOF. Since the dyadic doubly stochastic matrices form a dense subset of the space of all doubly stochastic matrices, it suffices to prove the lower bound for dyadic matrices only. Let $A = (a_{ij})_{i,j=1}^{n}$ be a dyadic doubly stochastic matrix. Let u be a natural number such that $2^u A$ is integral, and let for each $t \geq u$, G_t be the 2^t-regular bipartite graph with points $v_1,\dots,v_n,w_1,\dots,w_n$, where there are $2^t a_{ij}$ edges connecting v_i and w_j, for i,j = 1,...,n. This means that for $t \geq u$, the graph G_t arises from the graph G_u by replacing each edge by 2^{t-u} parallel edges.

Now the number μ of perfect matchings in G_u is easily seen to be equal to:

$$(62)\qquad \mu = 2^{un}.\mathrm{per}A.$$

Moreover, the number γ_t of 2^t-edge-colourings of G_t satisfies:

$$(63)\qquad \gamma_t \leq \mu^{2^t}.(2^{t-u})!^{2^u n},$$

since each colouring is determined by specifying 2^t perfect matchings in G_u, together with an ordering of the 2^{t-u} "copies" in G_t of each of the $2^u n$ edges of G_u. But by Corollary 7a we know:

$$(64)\qquad \gamma_t \geq \frac{(2^t)!^{2n}}{2^{t2^t n}}.$$

Combining (62), (63) and (64) gives a lower bound for perA depending on t and n, which, by Stirling's formula, tends to e^{-n} as $t\to\infty$. □

REMARK 2. Concluding we have met above the following upper and lower bounds for the functions f(k) and g(k).

$$(65)\qquad f(k) \leq \frac{(k-1)^{k-1}}{k^{k-2}},\quad g(k) \leq \frac{k!^2}{k^k},\quad f(1)=f(2)=g(1)=g(2)=1,\quad f(3)=\frac{4}{3},$$

$$f(k) \geq \frac{k}{e}, \quad g(k) \geq f(k)g(k-1) \geq f(k)f(k-1)\ldots f(1),$$

$$g(k\ell) \geq \left(\frac{(k\ell)!}{\ell!^k k!^\ell}\right)^2 . g(k)^\ell . g(\ell)^k$$

(cf. Theorem 4, 5, 6, Corollary 7b, (61)). Moreover, by methods similar to those for Theorem 7 one shows (cf. Valiant [28]):

(66) $$f(k\ell) \geq \binom{k\ell}{k}^{2/k} . \ell^{-2} . f(\ell) . g(k)^{1/k}.$$

[To prove this, we first show that each $k\ell$-regular bipartite graph G with 2n points has at least

(67) $$\left(\binom{k\ell}{k}^2 . \ell^{-2k} . f(\ell)^k\right)^n$$

k-factors. Indeed, make all possible graphs G' as in the proof of Theorem 7 (with $s = k$ and $t = \ell$). Each of these graphs has at least $f(\ell)^{kn}$ 1-factors. Each 1-factor of G' corresponds to a k-factor in G. Conversely, any fixed k-factor in G corresponds to a 1-factor in exactly

(68) $$\left(k! \frac{(k\ell-k)!}{(\ell-1)!^k}\right)^{2n}$$

graphs G' (the edges of the k-factor have to be divided among distinct points of G'). So the number of k-factors in G is at least

(69) $$\left(\frac{(k\ell)!}{\ell!^k}\right)^{2n} . f(\ell)^{kn} . \left(k! \frac{(k\ell-k)!}{(\ell-1)!^k}\right)^{-2n}$$

(using (59)), which is equal to (67).

Now we have:

(70) (the number of 1-factors in G)$^k \geq$ (the number of k-tuples of pairwise disjoint 1-factors in G) = (the number of pairs of a k-factor in G together with a k-edge-colouring of the k-factor) $\geq$ $\left(\binom{k\ell}{k}^2 . \ell^{-2k} . f(\ell)^k . g(k)\right)^n$,

which implies (66).]

Using the bounds of (65) and (66) one can derive the following bounds for f(k) and g(k) for k = 1,2,3,4,5,6,7,8,9,10:

$f(1)=1,$	$g(1)=1,$
$f(2)=1,$	$g(2)=1,$
$f(3)=\frac{4}{3},$	$g(3)=\frac{4}{3},$
$1.5 = \frac{3}{2} \leq f(4) \leq \frac{27}{16} = 1.6875,$	$g(4)=\frac{9}{4} = 2.25,$
$1.839 \approx 5/e \leq f(5) \leq \frac{256}{125} = 2.048,$	$4.139 \approx 45/4e \leq g(5) \leq 5!^2/5^5 = 4.608,$
$2.222 \approx 20/9 \leq f(6) \leq 5^5/6^4 \approx 2.411,$	$g(6) = 6!^2/6^6 \approx 11.111,$
$2.575 \approx 7/e \leq f(7) \leq 6^6/7^5 \approx 2.776,$	$28.613 \approx \frac{700}{9e} \leq g(7) \leq 7!^2/7^7 \approx 30.844,$
$2.943 \approx 8/e \leq f(8) \leq 7^7/8^6 \approx 3.142,$	$g(8) = 8!^2/8^8 \approx 96.899,$
$3.311 \approx 9/e \leq f(9) \leq 8^8/9^7 \approx 3.508,$	$g(9) = 9!^2/9^9 \approx 339.894,$
$3.679 \approx 10/e \leq f(10) \leq 9^9/10^8 \approx 3.874,$	$1250 \approx \frac{10g(9)}{e} \leq g(10) \leq 10!^2/10^{10} \approx 1316.819.$

5. BRÈGMAN'S UPPER BOUND.

It is easy to see that the *maximum* permanent of doubly stochastic matrices is 1. Similarly, the maximum permanent of matrices in Λ_n^k is k^n. However, if we go over to a further discretization, and we restrict the entries of the matrices to 0 and 1 only, less trivial upper bounds can be obtained. In 1963, Minc [20] published a conjectured upper bound (see Theorem 8 below), which was proved in 1973 by Brègman [4]. His proof is based on ideas from convex programming and on some theory of doubly stochastic matrices. Here we give the shorter proof as described in [23]. This proof uses the fact that if $t_1,\ldots,t_r$ are nonnegative real numbers, then:

$$(71) \qquad \left(\frac{t_1+\ldots+t_r}{r}\right)^{t_1+\ldots+t_r} \leq t_1^{t_1} \cdot \ldots \cdot t_r^{t_r}.$$

[This follows from the convexity of the function xlogx, by taking logarithms of both sides of (71), and by dividing these logarithms by r.]

<u>THEOREM 8</u> (Brègman's upper bound). *Let* A *be a square* {0,1}*-matrix of order* n, *with* r_i *ones in row* i $(1 \leq i \leq n)$. *Then*

$$(72) \qquad \mathrm{per}A \leq \prod_{i=1}^{n} r_i!^{1/r_i}.$$

<u>PROOF</u>. We use induction on n, the case n = 1 being trivial. Suppose the theorem has been shown for $(n-1)\times(n-1)$-matrices. We shall prove:

$$(73)\qquad (\mathrm{per}A)^{n\,\mathrm{per}A} \le \Big(\prod_{i=1}^{n} r_i!^{1/r_i}\Big)^{n\,\mathrm{per}A},$$

which implies (72).

We first give a series of (in)equalities, which we justify afterwards. The variables i, j and k range from 1 to n. Let S denote the set of all permutations σ of $\{1,\dots,n\}$ for which $a_{i\sigma(i)} = 1$ for all $i = 1,\dots,n$. So $|S| = \mathrm{per}A$.

$$(74)\qquad (\mathrm{per}A)^{n\,\mathrm{per}A} \overset{1}{=} \prod_i (\mathrm{per}A)^{\mathrm{per}A} \overset{2}{\le} \prod_i\Big(r_i^{\,\mathrm{per}A} \prod_{\substack{k\\ a_{ik}=1}} \mathrm{per}A_{ik}^{\,\mathrm{per}A_{ik}}\Big) =$$

$$\overset{3}{=} \prod_{\sigma\in S} \Big(\big(\prod_i r_i\big).\big(\prod_i \mathrm{per}A_{i\sigma(i)}\big)\Big) \le$$

$$\overset{4}{\le} \prod_{\sigma\in S} \Big(\big(\prod_i r_i\big).\big(\prod_i \big(\prod_{\substack{j\\ j\neq i\\ a_{j\sigma(i)}=0}} r_j!^{1/r_j}\big).\big(\prod_{\substack{j\\ j\neq i\\ a_{j\sigma(i)}=1}} (r_j-1)!^{1/r_j-1}\big)\big)\Big) =$$

$$\overset{5}{=} \prod_{\sigma\in S} \Big(\big(\prod_i r_i\big).\big(\prod_j \big(\prod_{\substack{i\\ i\neq j\\ a_{j\sigma(i)}=0}} r_j!^{1/r_j}\big).\big(\prod_{\substack{i\\ i\neq j\\ a_{j\sigma(i)}=1}} (r_j-1)!^{1/r_j-1}\big)\big)\Big) =$$

$$\overset{6}{=} \prod_{\sigma\in S} \Big(\big(\prod_i r_i\big).\big(\prod_j\big(r_j!^{(n-r_j)/r_j}.(r_j-1)!^{(r_j-1)/(r_j-1)}\big)\big)\Big) =$$

$$\overset{7}{=} \prod_{\sigma\in S} \Big(\prod_i r_i!^{n/r_i}\Big) \overset{8}{=} \Big(\prod_i r_i!^{1/r_i}\Big)^{n\,\mathrm{per}A}.$$

Explanation: [1] is trivial; [2] use (71) (note that r_i is the number of k such that $a_{ik}=1$ and $\mathrm{per}A = \sum_{k,a_{ik}=1}\mathrm{per}A_{ik}$); [3] the number of factors r_i equals perA on both sides, while the number of factors $\mathrm{per}A_{ik}$ equals the number of $\sigma \in S$ for which $\sigma(i) = k$ (this is $\mathrm{per}A_{ik}$ in case $a_{ik}=1$, and 0 otherwise); [4] apply the induction hypothesis to each $A_{i\sigma(i)}$ $(i = 1,\dots,n)$; [5] change the order of multiplication; [6] the number of i such that $i \neq j$ and $a_{j\sigma(i)}=0$ is $n-r_j$, while the number of i such that $i \neq j$ and $a_{j\sigma(i)}=1$ is r_j-1 (note that $a_{j\sigma(j)}=1$, and that the equality is proved for all fixed σ and j separately); [7] and [8] are trivial. □

In particular it follows that if all row sums of A are exactly k then

$$(75)\qquad \mathrm{per}A \le \big(k!^{1/k}\big)^n.$$

It is easy to see that for fixed k the ground number here is best possible, also if we restrict ourselves to {0,1}-matrices in Λ_n^k.

6. EULERIAN ORIENTATIONS.

As a further illustration of the results and methods above, we consider eulerian orientations. For any undirected graph G = (V,E), let $\varepsilon(G)$ denote the number of eulerian orientations of G. Here an *eulerian orientation* is an orientation of the edges such that at each vertex the indegree is equal to the outdegree.

Then if G is a loopless 2k-regular graph with n vertices, the number of eulerian orientations satisfies:

$$(76)\qquad \left(2^{-k}\binom{2k}{k}\right)^n \le \varepsilon(G) \le \left(\sqrt{\binom{2k}{k}}\right)^n,$$

and moreover, for each fixed k, the ground numbers in (76) cannot be improved ([25]).

There exists a direct relation between $\varepsilon(G)$ and the permanent function. Let G = (V,E) be a graph in which each vertex v has degree deg(v) even. Let B be the incidence matrix of G, with |V| rows and |E| columns. Let the matrix A arise from B by repeating, for each vertex v, the row of B corresponding with v ½deg(v) times. Then A is a square {0,1}-matrix of order |E|. Now one easily checks that:

$$(77)\qquad \varepsilon(G) = \frac{\text{per}A}{\prod_{v\in V}(\tfrac{1}{2}\deg(v))!}\ .$$

Substituting Brègman's upper bound (Theorem 8) in (77) gives:

$$(78)\qquad \varepsilon(G) \le \prod_{v\in V}\binom{\deg(v)}{\tfrac{1}{2}\deg(v)}^{\frac{1}{2}},$$

and the right hand side in (76) follows. The graph with 2 points connected by 2k parallel edges shows that we cannot have a lower ground number in the upper bound in (76).

Concerning lower bounds, Falikman and Egorychev's lower bound, in the form (6), gives that if G is 2k-regular, then $A\in\Lambda_{kn}^{2k}$, and so with (77):

$$(79)\qquad \varepsilon(G) \ge \left(\frac{2}{n}\right)^{kn}\cdot\frac{(nk)!}{k!^n}\ .$$

Asymptotically this implies:

(80) $$\varepsilon(G) \geq \left(\frac{1}{k!}\left(\frac{2k}{e}\right)^k\right)^n.$$

The conjecture (10) would imply the better lower bound:

(81) $$\varepsilon(G) \geq \left(\frac{1}{k!}\,\frac{(2k-1)^{2k-1}}{(2k)^{2k-2}}\right)^n.$$

However, the lower bound given in (76) is even higher (and is best possible). This is not surprising, as generally the permanent function seems to approach its minimum value if the matrix tends to have a random structure, whereas the matrix A obtained from G as above, has several equal rows.

The lower bound in (76) can be shown as follows. Let $\varepsilon(2d_1,\ldots,2d_n)$ be the minimum of $\varepsilon(G)$, where G ranges over all undirected graphs (possibly with loops) with degree sequence $2d_1,\ldots,2d_n$. Then:

(82) $$\varepsilon(2d_1,\ldots,2d_n) \geq \prod_{i=1}^{n} 2^{-d_i}\binom{2d_i}{d_i}.$$

This can be seen by induction on $2d_1+\ldots+2d_n$. If this sum is 0, (82) is trivial. If, say, $d_1 \geq 1$, let G be an undirected graph with degree sequence $2d_1,\ldots,2d_n$ and with $\varepsilon(G) = \varepsilon(2d_1,\ldots,2d_n)$. Let point v have degree $2d_1$, and let $e_1,\ldots,e_{2d_1}$ be the edges incident with v. For $1 \leq i < j \leq 2d_1$, let $\varepsilon_{ij}(G)$ denote the number of eulerian orientations of G in which e_i and e_j are oriented in series (i.e., one of them has v as tail, and the other has v as head). If, say, $e_i = \{u,v\}$ and $e_j = \{v,w\}$, let G_{ij} be the graph obtained from G by replacing e_i and e_j by one new edge $\{u,w\}$. Then:

(83) $$\varepsilon_{ij}(G) = \varepsilon(G_{ij}) \geq \varepsilon(2d_1-2,2d_2,\ldots,2d_n).$$

Therefore, inductively,

(84) $$\varepsilon(G) = \frac{1}{d_1^2}\sum_{1\leq i<j\leq 2d_1} \varepsilon_{ij}(G) \geq d_1^{-2}.\binom{2d_1}{2}\varepsilon(2d_1-2,2d_2,\ldots,2d_n) \geq$$
$$\geq d_1^{-2}\binom{2d_1}{2}2^{-(d_1-1)}\binom{2d_1-2}{d_1-1}\prod_{i=2}^{n} 2^{-d_i}\binom{2d_i}{d_i} = \prod_{i=1}^{n} 2^{-d_i}\binom{2d_i}{d_i}.$$

So (82) is proved, and the lower bound in (76) follows.

By averaging techniques, similar to those in the proofs of the Theorems

5 and 6, one shows that for fixed k the ground number in the lower bound in (76) is best possible. It is also best possible if we restrict G to loopless graphs. This follows with the help of the Alexandroff-Fenchel permanent inequality (Theorem 1) - see [25]. We conjecture that it is even best possible if G is restricted to simple graphs (i.e., no loops or multiple edges). Moreover, we conjecture that for simple graphs a better upper bound can be obtained: if G is a simple undirected graph with degree sequence $2d_1, \ldots, 2d_n$, then

$$(85) \qquad \text{(Conjecture)} \ \varepsilon(G) \leq \prod_{i=1}^{n} \varepsilon(K_{2d_i+1})^{1/(2d_i+1)}$$

(K_t being the complete undirected graph on t points). A problem we met in constructing a proof similar to that of Brègman's upper bound (Theorem 8) is that we could not find a suitable formula for $\varepsilon(K_t)$.

REFERENCES.

[1] A.D. Alexandroff, *K teorii smeshannykh ob'ëmov vypuklykh tel (Zur Theorie der gemischten Volumina von konvexen Körpern)*, Mat. Sb. 3 (45) (1938) 227-251 (Russian; German summary).

[2] T. Bang, *Matrixfunktioner som med et numerisk lille deficit viser v.d. Waerdens permanenthypothese*, preprint, 1978.

[3] G. Birkhoff, *Tres observaciones sobre el algebra lineal*, Rev. Univ. Nac. Tucuman Ser. A 5 (1946) 147-148.

[4] L.M. Brègman, *Some properties of nonnegative matrices and their permanents*, Soviet Math. Dokl. 14 (1973) 945-949 (English translation of: Dokl. Akad. Nauk SSSR 211 (1973) 27-30).

[5] H. Busemann, *Convex surfaces*, Interscience, New York, 1958.

[6] G.P. Egorychev, *Solution of van der Waerden's permanent conjecture*, Advances in Math. 42 (1981) 299-305 (English translation of preprint IFSO-13M of the Kirensko Institute of Physics, Krasnoiarsk, 1980).

[7] P. Erdös and A. Rényi, *On random matrices II*, Studia Sci. Math. Hungar. 3 (1968) 459-464.

[8] D.I. Falikman, *A proof of the van der Waerden conjecture on the permanent of a doubly stochastic matrix*, Mat. Zametki 29 (1981) 931-938 (Russian).

[9] W. Fenchel, *Inégalités quadratiques entre les volumes mixtes des corps convexes*, C.R. Acad. Sci. Paris 203 (1936) 647-650.

[10] S. Friedland, *A lower bound for the permanent of a doubly stochastic matrix*, Annals of Math. 110 (1979) 167-176.

[11] D.J. Hartfiel and J.W. Crosby, *On the permanent of a certain class of (0,1)-matrices*, Canad. Math. Bull. 14 (1971) 507-511.

[12] D.E. Knuth, *A permanent inequality*, Amer. Math. Monthly 88 (1981) 731-740.

[13] D. König, *Graphen und ihre Anwendung auf Determinantentheorie und Mengenlehre*, Math. Ann. 77 (1916) 453-465.

[14] D. König, *Über trennende Knotenpunkte in Graphen (nebst Anwendungen auf Determinanten und Matrizen)*, Acta Lit. Sci. Sect. Sci. Math. (Szeged) 6 (1932-1934) 155-179.

[15] J.H. van Lint, *Combinatorial Theory Seminar*, Springer Lecture Notes in Mathematics 382, Springer, Berlin, 1974.

[16] J.H. van Lint, *Notes on Egoritsjev's proof of the van der Waerden conjecture*, Linear Algebra and Appl. 39 (1981) 1-8.

[17] J.H. van Lint, *The van der Waerden conjecture: two proofs in one year*, Math. Intelligencer 4 (1982) 72-77.

[18] D. London, *Some notes on the van der Waerden conjecture*, Linear Algebra and Appl. 4 (1971) 155-160.

[19] M. Marcus and M. Newman, *On the minimum of the permanent of a doubly stochastic matrix*, Duke Math. J. 26 (1959) 61-72.

[20] H. Minc, *Upper bounds for permanents of (0,1)-matrices*, Bull. Amer. Math. Soc. 69 (1963) 789-791.

[21] H. Minc, *Permanents*, Encyclopedia of Mathematics and its Applications 6, Addison-Wesley, Reading, Mass., 1978.

[22] J. von Neumann, *A certain zero-sum two-person game equivalent to the optimum assignment problem*, in: "Contributions to the theory of games" (A.W. Tucker and H.W. Kuhn, eds.), Annals of Math. Studies

38, Princeton Univ. Press, Princeton, N.J., 1953, pp. 5-12.

[23] A. Schrijver, *A short proof of Minc's conjecture*, J. Combinatorial Theory (A) 25 (1978) 80-83.

[24] A. Schrijver, *On the number of edge-colourings of regular bipartite graphs*, Discrete Math. 38 (1982) 297-301.

[25] A. Schrijver, *Bounds on the number of eulerian orientations*, preprint, 1983.

[26] A. Schrijver and W.G. Valiant, *On lower bounds for permanents*, Indag. Math. 42 (1980) 425-427.

[27] L.G. Valiant, *The complexity of computing the permanent*, Theor. Comput. Sci. 8 (1979) 189-201.

[28] W.G. Valiant, *On an identity of T. Bang*, Report ZW 132/79, Mathematisch Centrum, Amsterdam, 1979.

[29] M. Voorhoeve, *A lower bound for the permanents of certain (0,1)-matrices*, Indag. Math. 41 (1979) 83-86.

[30] B.L. van der Waerden, *Aufgabe 45*, Jahresber. Deutsch. Math.-Verein. 25 (1926) 117.

[31] H.S. Wilf, *On the permanent of a doubly stochastic matrix*, Canad. J. Math. 18 (1966) 758-761.

ACKNOWLEDGEMENT

I like to thank Dr E. Keith Lloyd for his careful reading of the text and his many helpful comments.